PARTICLE-LADEN FLOWS: SOME CONUNDRUMS

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Some history

- Numerous papers (of variable quality) have been published
- Einstein (1906) investigated these flows
- An early investigation involving the stability of plane Poiseuille flow made by Saffman (1962)
- Michael (1968) investigated the (inviscid) dusty flow past a sphere
- *The Dynamics of Fluidized Particles* Jackson (2000)
(COMPREHENSIVE) EQUATIONS OF MOTION (A LA DREW, 1983)

Following Drew (1983)

$$\frac{\partial (\alpha_k \rho_k)}{\partial t} + \nabla \cdot (\alpha_k \rho_k \mathbf{v}_k) = 0$$

$$\frac{\partial (\alpha_k \rho_k \mathbf{v}_k)}{\partial t} + \nabla \cdot (\alpha_k \rho_k \mathbf{v}_k \mathbf{v}_k) = -\alpha_k \nabla p_k + \nabla \cdot \left( \alpha_k (\tau_k + \sigma_k) \right) + (p_{k,i} - p_k) \nabla \alpha_k + M_k$$

$k = 1$: particles, $k = 2$: fluid. $\tau_k$ is (approximately) stress tensor, $\sigma_k$ turbulent stress tensor, $p_{k,i}$ pressure at interface, $M_k$ ‘interfacial’ force density'; $\alpha, \rho, \mathbf{v}$ volume fraction, density and velocity fields.
Simplifications

- Turbulent stresses $\sigma_k = 0$
- Drew (1983) states $p_{k,i} = p_k$ for non-acoustic problems
- Drew (1983) states $p_1 = p_2 + p_c$, where $p_c$ pressure due to collisions; assume $p_1 = p_2$, and constant.
- Assume $\tau_1 = 0$ for solid particles
- Must have $\alpha_1 + \alpha_2 = 1$, $M_1 + M_2 = 0$
- Will consider 3 problems
NON-DIMENSIONAL EQUATIONS OF MOTION

\[ \frac{\partial \alpha}{\partial t} + \nabla \cdot ((1 - \alpha)u_f) = 0 \]

\[ \frac{\partial u_f}{\partial t} + (u_f \cdot \nabla) u_f = -\nabla p + \frac{1}{Re} \frac{1}{1 - \alpha} \nabla \cdot \left( (1 - \alpha)e \right) + \frac{\beta \alpha}{1 - \alpha} (u_p - u_f), \]

\[ \frac{\partial \alpha}{\partial t} + \nabla \cdot (\alpha u_p) = 0, \]

\[ \frac{\partial u_p}{\partial t} + (u_p \cdot \nabla) u_p = -\frac{1}{\gamma} \nabla p + \frac{\beta}{\gamma} (u_f - u_p) \]

Here \( Re = UL/\nu \), \( \beta = (9\nu L)/(2Ud^2) \) (Stokes drag), \( \gamma = \rho_p/\rho_f \), \( e \) rate of strain tensor for fluid.
Problem 1: steady dusty inviscid flow over a circular cylinder

Cylinder version of sphere problem considered by Michael (1968) - fluid affects particles but not v.v.

Polar coordinates \((r, \theta)\), velocity \((u, v)\), as \(\alpha \to 0\):

\[
(u_f(r, \theta), v_f(r, \theta))^T = \left( (1 - \frac{1}{r^2}) \cos \theta, -(1 + \frac{1}{r^2}) \sin \theta \right)^T
\]

Then \(\gamma \to \infty\) (heavy particles), \(\gamma/\beta = O(1)\):

\[
u_p \frac{\partial u_p}{\partial r} + \frac{v_p}{r} \frac{\partial u_p}{\partial \theta} - \frac{v_p^2}{r} = \frac{\beta}{\gamma} (u_f - u_p),
\]

\[
u_p \frac{\partial v_p}{\partial r} + \frac{v_p}{r} \frac{\partial v_p}{\partial \theta} + \frac{u_p v_p}{r} = \frac{\beta}{\gamma} (v_f - v_p),
\]

\[
\frac{1}{r} \frac{\partial}{\partial r} (r \alpha u_p) + \frac{1}{r} \frac{\partial}{\partial \theta} (\alpha v_p) = 0
\]
PARTICLE PATHS, $\beta/\gamma = 5$
‘Separation’ angle $\theta_{sep}$ for increasing values of inter-phase drag parameter $\beta/\gamma$. Note as $\beta/\gamma \to 0^+$, $\theta_{sep} \to \pi/2$ and as $\beta/\gamma \to 8^-$, $\theta_{sep} \to \pi$, as shown as dashed lines.
As \( r \to 1 \), can show \( u_p = u_{p0} + (r - 1)u_{p1} + \ldots \), where

\[
u_{p0} = 0, \quad u_{p1} = \frac{\beta}{2\gamma} \left(1 + \sqrt{1 - \frac{8\gamma}{\beta}}\right) \quad \text{for} \quad \frac{\beta}{\gamma} < 8,
\]

\[
u_{p0} \neq 0, \quad u_{p1} = -\frac{\beta}{\gamma} \quad \text{for} \quad \frac{\beta}{\gamma} > 8
\]
**Issues arising**

- Particles can ‘penetrate’ cylinder surface
- Solution discontinuous at $\beta/\gamma = 8$
- ‘Shadow’ regions
- Since on $\theta = \pi$, $r\alpha u_p = \text{constant}$, if $u_p \to 0$, $\alpha \to \infty$: violates $\alpha << 1$ condition
- A mixed elliptic/hyperbolic system
**Problem 2: settling under gravity**

Consider the stationary (1D) distribution of heavy ($\gamma \gg 1$) dust phase ‘settling’ under uniform gravity in an upwards propagating fluid; the dust-phase weight balanced by upwards motion of fluid. Fluid particle free and moving with constant speed $V_0$ for $y < 0$, $y = 0$ is location of stationary front.
INCLUDE (FLUID) VISCOSITY, AND (NOTIONALLY) ASSUME $\alpha = O(1)$

Problems of this type considered by Druzhinin (1994, 1995), with some inconsistencies. Particles affect fluid and v.v. Problem reduces to

$$V_p = 0$$

$$((1 - \alpha) V_f)' = 0$$

$$V_f V'_f + \frac{V_f}{1 - \alpha} = \gamma - 1 + \frac{2}{Re} V''_f - \frac{2}{Re} \frac{\alpha' V'_f}{1 - \alpha}$$

Assume $V_f(y = 0) = V_0$ (constant) and $\alpha(y = 0) = 0$. 
(BASEFLOW) RESULTS FOR $V(y)$ AND $\alpha(y)$

(a): $\gamma = 2$, $V_0 = 0.5$, $Re = 1, 2, 4, 8, 16, 32$ (solid lines), the dashed lines show the $Re \gg 1$ solution, (b): $V_0 = 1.5$, $Re = 20$, $\gamma = 4, 8, 16, 32$ and (c): $\gamma = 4$, $Re = 20$, $V_0 = 0.5, 1, 1.5, 2, 2.5$
Can perturb the inviscid system for a steady base flow via

\[ \alpha = \alpha_B(y) + \epsilon \tilde{\alpha}(y) e^{-i\omega t}, \]
\[ V_f = V_{fB}(y) + \epsilon \tilde{v}_f(y) e^{-i\omega t}, \]
\[ V_p = 0 + \epsilon \tilde{v}_p(y) e^{-i\omega t}, \]

Here \( \epsilon \ll 1 \), and \( \omega \) is a real frequency.

Focus on \( y \to \infty \):

\[ \alpha_B \to \alpha_\infty = 1 - \sqrt{\frac{V_0}{\gamma - 1}} \]
\[ V_f \to V_\infty = ((\gamma - 1) V_0)^{\frac{1}{2}} \]
Further supposing \((\tilde{\alpha}(y), \tilde{v}_f(y), \tilde{v}_p(y)) = (\hat{\alpha}, \hat{v}_f, \hat{v}_p)e^{iky}\) Then

\[
k^2 + k \left( -\frac{2\omega}{V_\infty} + \frac{2}{iV_\infty(1 - \alpha_\infty)} \right) + \left( \frac{\omega^2(\gamma + \alpha_\infty(1 - \gamma))}{\alpha_\infty V_\infty^2} - \frac{\omega}{iV_\infty^2\alpha_\infty(1 - \alpha_\infty)} \right) = 0
\]

Considering the limit \(\omega \to \infty\), we write \(k = \omega K, K = O(1)\),

\[
K = \frac{1}{V_\infty} \pm \frac{i}{V_\infty} \sqrt{\frac{\gamma(1 - \alpha)}{\alpha_\infty}}
\]

Spatial growth is

\[
\exp \left( \frac{\omega \gamma^{\frac{1}{2}}}{(\gamma - 1)^{\frac{3}{4}} V_0^{\frac{1}{4}}} \left( 1 - \left( \frac{V_0}{(\gamma - 1)} \right)^{\frac{1}{2}} \right)^{-\frac{1}{2}} y \right)
\]

Spatial growth rate proportional to frequency - implies problem ill-posed.
### Linear Stability, Finite $Re$

Equation for $k$ now a cubic:

$$k^3 \left( \frac{2\alpha_\infty V_\infty}{(1 - \alpha_\infty)\omega Re} \right) + k^2 \left( \frac{iV_\infty^2 \alpha_\infty}{\omega(1 - \alpha_\infty)} - \frac{2\alpha_\infty}{Re(1 - \alpha_\infty)} \right)$$

$$+ k \left( \frac{2\alpha_\infty V_\infty}{(1 - \alpha_\infty)^2 \omega} - \frac{2iV_\infty \alpha_\infty}{1 - \alpha_\infty} \right) + \left( i\omega \left( \gamma + \frac{\alpha_\infty}{1 - \alpha_\infty} \right) - \frac{1}{(1 - \alpha_\infty)^2} \right) = 0$$

As $\omega \to \infty$, $Re = O(1)$, two families:-

$$k \to \pm \left( -\frac{i\omega \gamma(1 - \alpha_\infty) + \alpha_\infty)Re}{2\alpha_\infty} \right)^{\frac{1}{2}}$$

$$k \to \frac{\omega}{V_\infty} - \frac{i\gamma V_\infty(1 - \alpha_\infty)Re}{2\alpha_\infty}$$
Problem 3: Boundary layers in a dilute particle suspension

Foster, Duck & Hewitt (2006) - mixed parabolic/hyperbolic problem

\[ u_e \sim x^m \]

\[ |2\theta| = \frac{2m\pi}{m+1} \]

Flow geometry appropriate for Falkner–Skan-type edge conditions, although solutions not restricted to have self similarity. Assume that local gravitational forcing is aligned as shown and thus the upper boundary layer is such that \( K > 0 \) whilst the lower boundary layer has \( K < 0 \), where

\[ K = \frac{gLRe^{1/2}}{U_\infty^2} (1 - \frac{1}{\gamma}) \]
DUSTY BOUNDARY-LAYER EQUATIONS

Usual boundary-layer scalings

\[ uu_x + vu_y + \bar{p}_x = u_{yy} - \beta\alpha(u - u_p), \]
\[ u_p u_{px} + v_p u_{py} = \frac{\beta}{\gamma}(u - u_p), \]
\[ u_p v_{px} + v_p v_{py} = \frac{\beta}{\gamma}(v - v_p) - K \cos \theta, \]
\[ u_x + v_y = 0, \]
\[ u_p \alpha_x + v_p \alpha_y = -\alpha(u_{px} + v_{py}). \]

\[ u = v = 0 \text{ on } y = 0, \quad u \to u_e(x) \text{ as } y \to \infty \]

Choice of boundary conditions for the particle phase is somewhat subtle; notionally

\[ u_p \to u_{pe}(x), \quad \text{and} \quad \alpha \to \alpha_e(x), \quad \text{for} \quad y \to \infty. \]

Will consider \( U_e(x) = x^m, \; 0 \leq m \leq 1 \).
THE OUTER FLOW AND CONDITIONS AT THE BOUNDARY-LAYER EDGE

\[ u_{pe}u'_{pe} = \frac{\beta}{\gamma} (u_e - u_{pe}), \quad u_{pe}E'_e + E^2_e + \frac{\beta}{\gamma} E_e = -\frac{\beta}{\gamma} u'_e, \quad u_{pe}\alpha'_e + \alpha_e D_e = 0, \quad (1) \]

where \( E_e(x) = \frac{\partial v_p}{\partial y} (y \to \infty) \), \( u_{pe}(x) = u_p (y \to \infty) \) and \( D_e(x) = D (y \to \infty) = u'_{pe} + E_e \) are the relevant functions evaluated as \( y \to \infty \). Also

\[ u_{pe}D'_e + \frac{\beta}{\gamma} D_e + (u'_{pe})^2 + E^2_e = 0. \]

\[ u_{pe} \left[ u'_{pe} + \frac{\beta}{\gamma} \right] = \frac{\beta}{\gamma} u_e, \]

\[ \left( u_{pe} \frac{d}{dx} + \frac{\beta}{\gamma} \right) \left( u_{pe} \frac{\alpha'_e}{\alpha_e} \right) = (u'_{pe})^2 + E^2_e. \]

For given fluid edge behaviour \( u_e(x) \), can determine streamwise particle motion \( u_{pe}(x) \), then particle motion normal to boundary \( E_e \), and finally external volume fraction \( \alpha_e \).
Development of the edge quantities (a) $u_{pe}$ and (b) $\alpha_e$; solid $m = .50$; dashed $m = .211$; dotted $m = .10$, all with $\beta/\gamma = 1$; $K$ not relevant here.

Can show $\alpha_e \sim \frac{1}{x-x_0}$ if $m > m_{crit}$ - violates $\alpha << 1$
Development of wall values with $x$ for $m = 0$, $\beta/\gamma = 1$. In case (b), leading-order asymptotic forms for $x \to \infty$ are shown for $x > 4.5$;
Taking $\kappa = 0$ (for simplicity - particle flow can be solved explicitly on $y = 0$):

$$\alpha_w = \frac{\alpha_0}{U_{pw}} = \frac{\alpha_0}{1 - \frac{\beta x}{\gamma}}, \quad U_{pw} > 0.$$  

Therefore volume fraction is singular part way along wall, and so model breaks down (Wang & Glass, 1988 continued their computation through the singularity).

Can be a ‘race’ between inner and outer singularities.
If gravitational forces act away from the wall, close to wall characteristics directed outwards; local analysis (as $x \to 0$) reveals a discontinuity in $\alpha$ along $y = y_{\text{crit}} = -\mathcal{K} \cos \theta / U_{p0}$ where $u_{p} = U_{p0} + \ldots$:

for $y < y_{\text{crit}}$, $\alpha = 0$ (particle free), for $y > y_{\text{crit}}$, $\alpha = \alpha_{0}$
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- Computations and analyses inform each other
- Contaminants can have a profound effect!