What is a flux?

Finite Volume methods (and others)

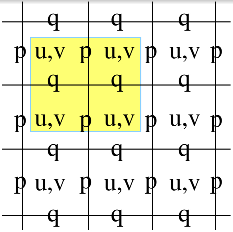
(are based on ensuring conservation by computing the flux through the surfaces of a polyhedral box. Either the normal component of the flux is evaluated at the center of the face, or at the collocation points of some quadrature rule.

Consider however the solution of the two-dimensional acoustic equation (successor to the ICASE equation?)

$$\begin{aligned} \partial_t p + \nabla \cdot \mathbf{u} &= 0 \\ \partial_t \mathbf{u} + \nabla p &= 0 \end{aligned}$$

An important property of the exact solution is that $\partial_t \nabla \times \mathbf{u} = 0$. However, vorticity arises from truncation error in almost all finite-volume schemes.

Spurious vorticity is generated numerically



Here u, v are velocities in the cells, and are updated by pressures (fluxes) p that have been computed on the vertical faces, and q on the horizontal faces

$$\delta_t u = \delta_x p, \qquad \delta_t v = \delta_y q$$

The vorticity evaluated at a vertex as $\omega = \mu_y \delta_x v - \mu_x \delta_y u$ will change by

$$\delta_t \omega = \mu_y \delta_x \delta_t v - \mu_x \delta_y \delta_t u = \mu_y \delta_x \delta_y q - \mu_x \delta_y \delta_x p$$

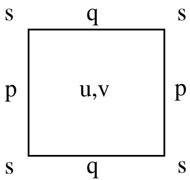
$$= \delta_x \delta_y (\mu_y q - \mu_x p)$$

so that vorticity is preserved only if $\mu_y q = \mu_x p$, implying that

$$p = \mu_y s, \qquad q = \mu_x s$$

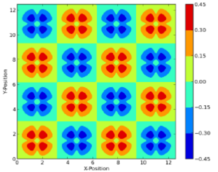
for some <u>vertex</u> flux s. Morton and Roe (SISC, 2001)

Face-based fluxes cannot preserve vorticity

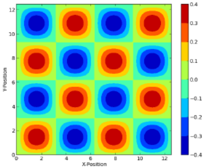


unless they are derived from vertex fluxes. These arguments were confirmed and extended by Mishra and Tadmor (SINUM 2011).

The experiments show patterns of vorticity in the initial data being disrupted by acoustic waves when using the standard MUSCL scheme, but preserved by a scheme based on corner fluxes.



Compact Vorticity Contours, MUSCL-H, t



Compact Vorticity Contours, FCT, t=3

Calculating a vertex flux involves averaging

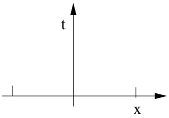
in the transverse direction. If the face flux for a conventional second order scheme reads

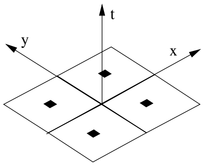
$$\mathbf{f} = \mu\mathbf{f} - \tfrac{1}{2}\mathbf{Q}\delta\mathbf{u}$$

the formula for a vertex flux reads

$$\mathbf{f} = \mu_x\mu_y\mathbf{f} - \tfrac{1}{2}\left\{\mathbf{Q}_x\mu_y\delta_x\mathbf{u} + \mathbf{Q}_y\mu_x\delta_y\mathbf{u}\right\}$$

The averaging operators serve to enhance the accuracy of Lax-Wendroff type schemes, particularly for waves not travelling in the grid directions. (Morton and Roe, 2001)





For the acoustic equations

the dissipative terms are physically meaningful

$$\mathbf{f} = \begin{pmatrix} u \\ v \\ p \end{pmatrix} = \begin{pmatrix} \mu_x \mu_y u - \frac{1}{2} Q \mu_y \delta_x p \\ \mu_x \mu_y v - \frac{1}{2} Q \mu_x \delta_x p \\ \mu_x \mu_y p - \frac{1}{2} Q(\mu_y \delta_x u + \mu_x \delta_y v) \end{pmatrix} \quad (1)$$

where $Q = 1.0$ **gives an upwind scheme, and** $Q = \nu$ **gives Lax-Wendroff. The pressure gradient and the divergence are the drivers of change. These drivers need to be subjected to some nonlinear mechanism that avoids oscillations. The key is to select an appropriate value of** Q**.**

Flux-corrected Transport (FCT)

provides an alternative to MUSCL-type schemes. It is widely regarded as a blunt instrument in comparison, but can be implemented in more sophisticated ways than it has been. The main idea is to represent all fluxes as the average of a high-order and a low-order term.

$$\mathbf{f} = \alpha \mathbf{f}_{HO} + (1 - \alpha)\mathbf{f}_{LO}$$

Begin by computing a low-order solution $\mathbf{u}^*$. In cells containing a local extremum, it is acceptable to proceed with a high-order correction, but for other cells the correction is limited;

$$\mathbf{f} = \mathbf{f}_{LO} + \alpha(\mathbf{f}_{HO} - \mathbf{f}_{LO})$$

where α is chosen to avoid an extremum in any neighboring cell.

Potential advantages of FCT are

that it is not based on one-dimensional physics and can be applied to invariant quantities like divergence and vorticity

These are new possibilities that have not been exploited. The monotonicity constraints that have been used in the past have been rather blunt instruments, which has led to FCT having a poor reputation in aerospace circles.

Is operator splitting OK?

Operator splitting is another idea with a bad reputation. Even if the individual operators were to be solved perfectly, there a splitting error unless the operators commute. The splitting into advection terms and acoustic terms has no such error (at the linear level)

$$\partial_t \begin{pmatrix} \rho \\ \rho\vec{v} \\ E \end{pmatrix} = \mathrm{div} \begin{pmatrix} \rho\vec{v} \\ \rho\vec{v} \otimes \vec{v} \\ E\vec{v} \end{pmatrix} + \mathrm{div} \begin{pmatrix} 0 \\ \mathbf{I}p \\ p\vec{v} \end{pmatrix}$$

$$\mathbf{u}^{n+1} = (advection\ (\Delta t/2))\ (Acoustic\ (\Delta t))\ (advection(\Delta t/2))\mathbf{u}^n$$

Only the advection operator needs to be upwinded. Both operators require limiting.

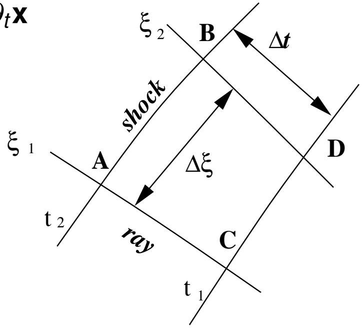
Geometrical Shock Dynamics (Whitham, 1957)

is a semiempiriical method for computing the evolution of a shock surface moving into a uniform (or stationary) flow. Lines that are everywhere normal to the shock are called rays. and bundles of them are raytubes. The speed of the shock normal to itself is a function of the area of the raytube. The distance between two points lying on the shock at different times is $d\mathbf{x} = \mathbf{m}dt + \mathbf{g}d\xi$, where $\mathbf{g} = \partial_\xi \mathbf{x}, \mathbf{m} = \partial_t \mathbf{x}$

There is a geometric conservation law

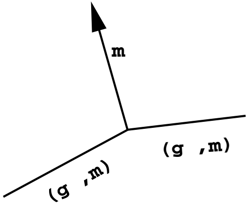
$$\partial_t \mathbf{g} + \partial_\xi \mathbf{m} = 0$$

closed by a relationship $m/m_0 = fn(g/g_0)$



The Riemann problem in two dimensions

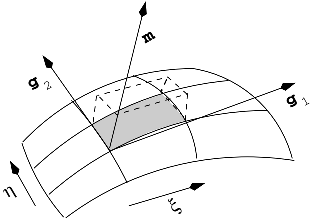
has data comprising two consecutive segments $\mathbf{g}_L, \mathbf{g}_R$ of the shock front. The solution is m, the speed and direction of the ray emanating from their juncture.

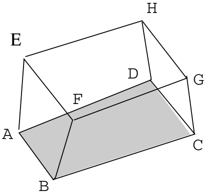


$$\mathbf{m} = \tfrac{1}{2}(\mathbf{m}_L + \mathbf{m}_R) - \tfrac{1}{2}\lambda(\mathbf{g}_R - \mathbf{g}_L)$$

where $\lambda = \sqrt{\frac{m(g)m'(g)}{g}}$ is the speed with which disturbances travel along the front. When all of the ray segments mΔt are in place, the new shock segments are defined.

Geometrical shock dynamics in three dimensions





is based on a distance

$$d\mathbf{x} = \mathbf{m}\ dt + \mathbf{g}_1\ d\xi + \mathbf{g}_2\ d\eta$$

and a GCL (floor + ceiling + sum of sides =0)

$$\partial_t(\mathbf{g}_1 \times \mathbf{g}_2) + \partial_\xi(\mathbf{m} \times \mathbf{g}_1) + \partial_\eta(\mathbf{m} \times \mathbf{g}_2) = 0$$

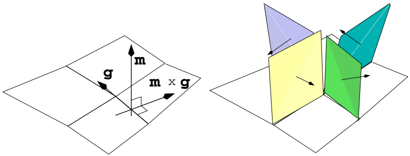
together with a closure

$$\mathbf{m}/m_0 = fn((\mathbf{g}_1 \times \mathbf{g}_2)/g_0)$$

Following classical procedure

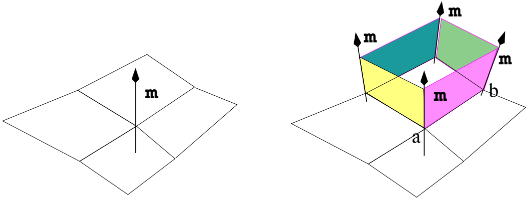
we consider pairs of shock patches and solve the 1D Riemann problem at their common edge. The output from this is the vector normal to the 'wall' based on that edge. The walls erected around one vertex do not meet in a line, but each wall could be twisted by an arbitrary amount. In order to force them all into alignment we require three constraints per vertex.

Since we have only one degree of freedom per side the discrepancies cannot be reconciled. This difficulty is currently encountered when introducing Riemann solvers into Lagrangian methods



Solving two-dimensional Riemann problems instead

results in a unique vector m leaving each vertex



Again, use of the vertex flux has allowed a multidimensional constaint (closure of the patch) to be met.