PARTICLE-LADEN FLOWS: SOME CONUNDRUMS

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Some history

- Numerous papers (of variable quality) have been published
- Annual Review articles by Marple (1970) and Drew (1983)
- Einstein (1906) investigated these flows
- An early investigation involving the stability of plane Poiseuille flow made by Saffman (1962)
- Michael (1968) investigated the (inviscid) dusty flow past a sphere
- *Hydrodynamics of Suspensions* Ungarish (1993)
- *The Dynamics of Fluidized Particles* Jackson (2000)

(COMPREHENSIVE) EQUATIONS OF MOTION (A LA DREW, 1983)

Following Drew (1983)

$$\frac{\partial(\alpha_k \rho_k)}{\partial t} + \nabla \cdot (\alpha_k \rho_k \underline{v_k}) = 0$$

$$\begin{aligned}\frac{\partial(\alpha_k \rho_k \underline{v_k})}{\partial t} + \nabla \cdot (\alpha_k \rho_k \underline{v_k v_k}) &= -\alpha_k \nabla p_k + \nabla \cdot \left(\alpha_k (\underline{\underline{\tau_k}} + \underline{\underline{\sigma_k}}) \right) \\ &+ (p_{k,i} - p_k) \nabla \alpha_k + M_k\end{aligned}$$

$k = 1$: particles, $k = 2$: fluid. $\underline{\underline{\tau_k}}$ is (approximately) stress tensor, $\underline{\underline{\sigma_k}}$ turbulent stress tensor, $p_{k,i}$ pressure at interface, M_k 'interfacial' force density'; α, ρ, $\underline{v}$ volume fraction, density and velocity fields.

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Simplifications

- Turbulent stresses $\underline{\underline{\sigma_k}} = 0$
- Drew (1983) states $p_{k,i} = p_k$ for non-acoustic problems
- Drew (1983) states $p_1 = p_2 + p_c$, where p_c pressure due to collisions; assume $p_1 = p_2$, and constant.
- Assume $\underline{\underline{\tau_1}} = 0$ for solid particles
- Must have $\alpha_1 + \alpha_2 = 1$, $M_1 + M_2 = 0$
- Will consider 3 problems

Non-dimensional equations of motion

$\underline{u_f}$, $\underline{u_p}$ fluid, particle velocities

$$-\frac{\partial\alpha}{\partial t} + \nabla\cdot\left((1-\alpha)\underline{u_f}\right) = 0$$

$$\frac{\partial \underline{u_f}}{\partial t} + (\underline{u_f}\cdot\nabla)\underline{u_f} = -\nabla p + \frac{1}{Re}\frac{1}{1-\alpha}\nabla\cdot\Big((1-\alpha)\underline{\underline{e}}\Big) + \frac{\beta\alpha}{1-\alpha}(\underline{u_p}-\underline{u_f}),$$

$$\frac{\partial\alpha}{\partial t} + \nabla\cdot(\alpha\underline{u_p}) = 0,$$

$$\frac{\partial \underline{u_p}}{\partial t} + (\underline{u_p}\cdot\nabla)\underline{u_p} = -\frac{1}{\gamma}\nabla p + \frac{\beta}{\gamma}(\underline{u_f}-\underline{u_p})$$

Here $Re = UL/\nu$, $\beta = (9\nu L)/(2Ud^2)$ (Stokes drag), $\gamma = \rho_p/\rho_f$, $\underline{\underline{e}}$ rate of strain tensor for fluid.

PROBLEM 1: STEADY DUSTY INVISCID FLOW OVER A CIRCULAR CYLINDER

Cylinder version of sphere problem considered by Michael (1968) - fluid affects particles but not v.v.
Polar coordinates (r, θ), velocity (u, v), as $\alpha \to 0$:

$$(u_f(r,\theta), v_f(r,\theta))^T = \left((1 - \frac{1}{r^2})\cos\theta, -(1 + \frac{1}{r^2})\sin\theta\right)^T$$

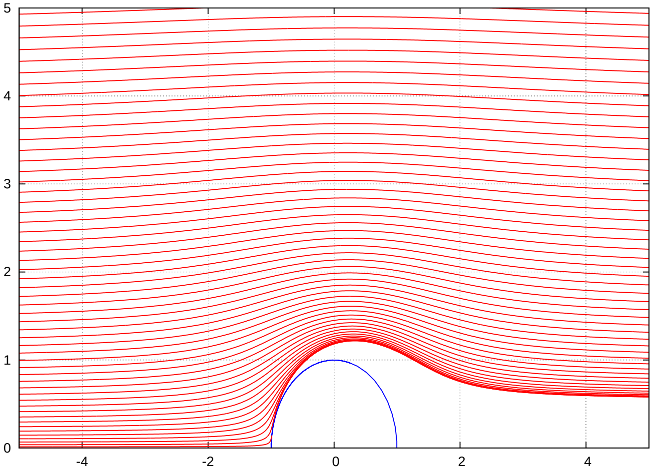
Then $\gamma \to \infty$ (heavy particles), $\gamma/\beta = O(1)$:

$$u_p \frac{\partial u_p}{\partial r} + \frac{v_p}{r}\frac{\partial u_p}{\partial \theta} - \frac{v_p^2}{r} = \frac{\beta}{\gamma}(u_f - u_p),$$

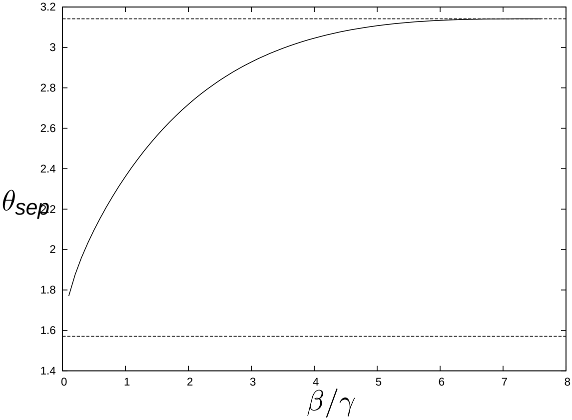
$$u_p \frac{\partial v_p}{\partial r} + \frac{v_p}{r}\frac{\partial v_p}{\partial \theta} + \frac{u_p v_p}{r} = \frac{\beta}{\gamma}(v_f - v_p),$$

$$\frac{1}{r}\frac{\partial}{\partial r}(r\alpha u_p) + \frac{1}{r}\frac{\partial}{\partial \theta}(\alpha v_p) = 0$$

Particle paths, $\beta/\gamma = 5$



SEPARATION ANGLE



'Separation' angle θ_{sep} for increasing values of inter-phase drag parameter β/γ.. Note as $\beta/\gamma \to 0^+$, $\theta_{sep} \to \pi/2$ and as $\beta/\gamma \to 8^-$, $\theta_{sep} \to \pi$, as shown as dashed lines

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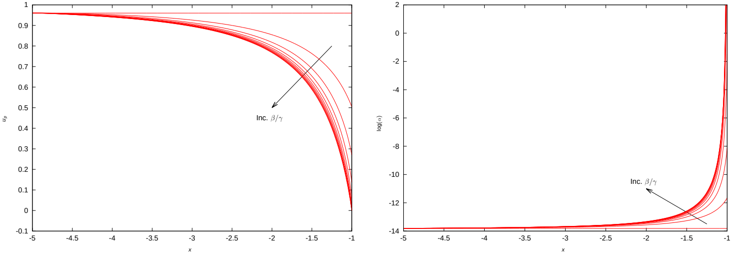
Conditions along $\theta = \pi$



As $r \to 1$, can show $u_p = u_{p0} + (r-1)u_{p1} + \ldots$, where

$$u_{p0} = 0, \quad u_{p1} = \frac{\beta}{2\gamma}\left(1 + \sqrt{1 - \frac{8\gamma}{\beta}}\right) \quad \text{for} \quad \beta/\gamma < 8,$$

$$u_{p0} \neq 0, \quad u_{p1} = -\beta/\gamma \quad \text{for} \quad \beta/\gamma > 8$$

Issues arising

- Particles can 'penetrate' cylinder surface
- Solution discontinuous at $\beta/\gamma = 8$
- 'Shadow' regions
- Since on $\theta = \pi$, $r\alpha u_p =$ constant, if $u_p \to 0$, $\alpha \to \infty$: violates $\alpha << 1$ condition
- A mixed elliptic/hyperbolic system

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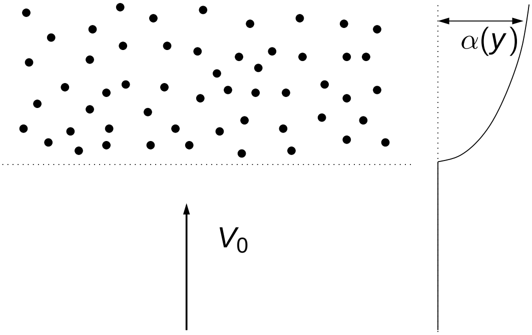
Problem 2: settling under gravity

Consider the stationary (1D) distribution of heavy ($\gamma >> 1$) dust phase 'settling' under uniform gravity in an upwards propagating fluid; the dust-phase weight balanced by upwards motion of fluid. Fluid particle free and moving with constant speed V_0 for $y < 0$, $y = 0$ is location of stationary front.



INCLUDE (FLUID) VISCOSITY, AND (NOTIONALLY) ASSUME $\alpha = O(1)$)

.
Problems of this type considered by Druzhinin (1994, 1995), with some inconsistencies.
Particles affect fluid and v.v. Problem reduces to

$$V_p = 0$$

$$((1-\alpha)V_f)' = 0$$

$$V_f V_f' + \frac{V_f}{1-\alpha} = \gamma - 1 + \frac{2}{Re} V_f'' - \frac{2}{Re} \frac{\alpha' V_f'}{1-\alpha}$$

Assume $V_f(y=0) = V_0$ (constant) and $\alpha(y=0) = 0$.

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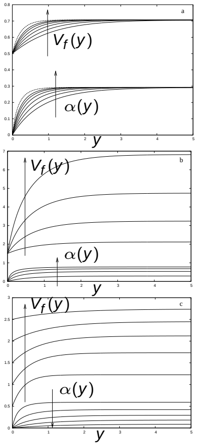
(BASEFLOW) RESULTS FOR $V(y)$ AND $\alpha(y)$



(a): $\gamma = 2$, $V_0 = 0.5$, $Re = 1, 2, 4, 8, 16, 32$ (solid lines), the dashed lines show the $Re \gg 1$ solution, (b): $V_0 = 1.5$, $Re = 20$, $\gamma = 4, 8, 16, 32$ and (c): $\gamma = 4$, $Re = 20$, $V_0 = 0.5, 1, 1.5, 2, 2.5$

SPATIAL LINEAR STABILITY IN THE INVISCID LIMIT

Can perturb the inviscid system for a steady base flow via

$$\alpha = \alpha_B(y) + \epsilon\tilde{\alpha}(y)e^{-i\omega t},$$

$$V_f = V_{fB}(y) + \epsilon\tilde{v}_f(y)e^{-i\omega t},$$

$$V_p = 0 + \epsilon\tilde{v}_p(y)e^{-i\omega t},$$

Here $\epsilon << 1$, and ω is a real frequency.
Focus on $y \to \infty$:

$$\alpha_B \to \alpha_\infty = 1 - \sqrt{\frac{V_0}{\gamma - 1}}$$

$$V_f \to V_\infty = ((\gamma - 1)V_0)^{\frac{1}{2}}$$

LINEAR STABILITY IN THE INVISCID LIMIT (CONTINUED)

Further supposing $(\tilde{\alpha}(y), \tilde{v}_f(y), \tilde{v}_p(y)) = (\hat{\alpha}, \hat{v}_f, \hat{v}_p)e^{iky}$ Then

$$k^2 + k\left(-\frac{2\omega}{V_\infty} + \frac{2}{iV_\infty(1-\alpha_\infty)}\right) + \left(\frac{\omega^2(\gamma + \alpha_\infty(1-\gamma))}{\alpha_\infty V_\infty^2} - \frac{\omega}{iV_\infty^2\alpha_\infty(1-\alpha_\infty)}\right) = 0$$

Considering the limit $\omega \to \infty$, we write $k = \omega K$, $K = O(1)$,

$$K = \frac{1}{V_\infty} \pm \frac{i}{v_\infty}\sqrt{\frac{\gamma(1-\alpha)}{\alpha_\infty}}$$

Spatial growth is

$$\exp\left(\frac{\omega\gamma^{\frac{1}{2}}}{(\gamma-1)^{\frac{3}{4}}V_0^{\frac{1}{4}}}\left(1 - \left(\frac{V_0}{(\gamma-1)}\right)^{\frac{1}{2}}\right)^{-\frac{1}{2}} y\right)$$

Spatial growth rate proportional to frequency - implies problem ill-posed.

LINEAR STABILITY, FINITE Re

Equation for k now a cubic:

$$k^3\left(\frac{2\alpha_\infty V_\infty}{(1-\alpha_\infty)\omega Re}\right)+k^2\left(\frac{iV_\infty^2\alpha_\infty}{\omega(1-\alpha_\infty)}-\frac{2\alpha_\infty}{Re(1-\alpha_\infty)}\right)$$
$$+k\left(\frac{2\alpha_\infty V_\infty}{(1-\alpha_\infty)^2\omega}-\frac{2iV_\infty\alpha_\infty}{1-\alpha_\infty}\right)+\left(i\omega\left(\gamma+\frac{\alpha_\infty}{1-\alpha_\infty}\right)-\frac{1}{(1-\alpha_\infty)^2}\right)=0$$

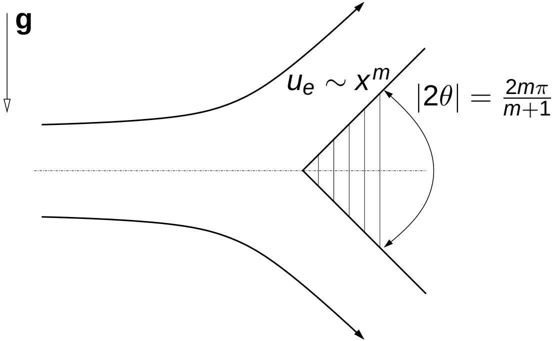
As $\omega\to\infty$, $Re=O(1)$, two families:-

$$k\to\pm\left(\frac{-(i\omega\gamma(1-\alpha_\infty)+\alpha_\infty)Re}{2\alpha_\infty}\right)^{\frac{1}{2}}$$

$$k\to\frac{\omega}{V_\infty}-\frac{i\gamma V_\infty(1-\alpha_\infty)Re}{2\alpha_\infty}$$

PROBLEM 3: BOUNDARY LAYERS IN A DILUTE PARTICLE SUSPENSION

Foster, Duck & Hewitt (2006) - mixed parabolic/hyperbolic problem



Flow geometry appropriate for Falkner–Skan-type edge conditions, although solutions not restricted to have self similarity. Assume that local gravitational forcing is aligned as shown and thus the upper boundary layer is such that $\mathcal{K} > 0$ whilst the lower boundary layer has $\mathcal{K} < 0$, where $\mathcal{K} = \frac{gLRe^{1/2}}{U_\infty^2}(1 - \frac{1}{\gamma})$

DUSTY BOUNDARY-LAYER EQUATIONS

Usual boundary-layer scalings

$$\begin{aligned}
uu_x + vu_y + \bar{p}_x &= u_{yy} - \beta\alpha(u - u_p), \\
u_p u_{px} + v_p u_{py} &= \frac{\beta}{\gamma}(u - u_p), \\
u_p v_{px} + v_p v_{py} &= \frac{\beta}{\gamma}(v - v_p) - \mathcal{K}\cos\theta, \\
u_x + v_y &= 0, \\
u_p \alpha_x + v_p \alpha_y &= -\alpha(u_{px} + v_{py}).
\end{aligned}$$

$$u = v = 0 \quad \text{on} \quad y = 0, \quad u \to u_e(x) \quad \text{as} \quad y \to \infty$$

Choice of boundary conditions for the particle phase is somewhat subtle; notionally

$$u_p \to u_{pe}(x), \quad \text{and} \quad \alpha \to \alpha_e(x), \quad \text{for} \quad y \to \infty.$$

Will consider $U_e(x) = x^m$, $0 \le m \le 1$

THE OUTER FLOW AND CONDITIONS AT THE BOUNDARY-LAYER EDGE

$$u_{pe}u'_{pe} = \frac{\beta}{\gamma}(u_e - u_{pe}), \quad u_{pe}E'_e + E_e^2 + \frac{\beta}{\gamma}E_e = -\frac{\beta}{\gamma}u'_e, \quad u_{pe}\alpha'_e + \alpha_e\mathcal{D}_e = 0, \qquad (1)$$

where $E_e(x) = \partial v_p/\partial y(y \to \infty)$, $u_{pe}(x) = u_p(y \to \infty)$ and $\mathcal{D}_e(x) = \mathcal{D}(y \to \infty) = u'_{pe} + E_e$ are the relevant functions evaluated as $y \to \infty$ Also

$$u_{pe}\mathcal{D}'_e + \frac{\beta}{\gamma}\mathcal{D}_e + (u'_{pe})^2 + E_e^2 = 0.$$

$$u_{pe}\left[u'_{pe} + \frac{\beta}{\gamma}\right] = \frac{\beta}{\gamma}u_e,$$

$$\left(u_{pe}\frac{d}{dx} + \frac{\beta}{\gamma}\right)\left(u_{pe}\frac{\alpha'_e}{\alpha_e}\right) = (u'_{pe})^2 + E_e^2.$$

For given fluid edge behaviour $u_e(x)$, can determine streamwise particle motion $u_{pe}(x)$, then particle motion normal to boundary E_e, and finally external volume fraction $\alpha_e(x)$.

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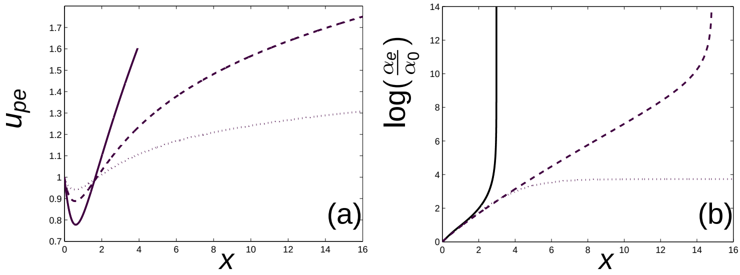
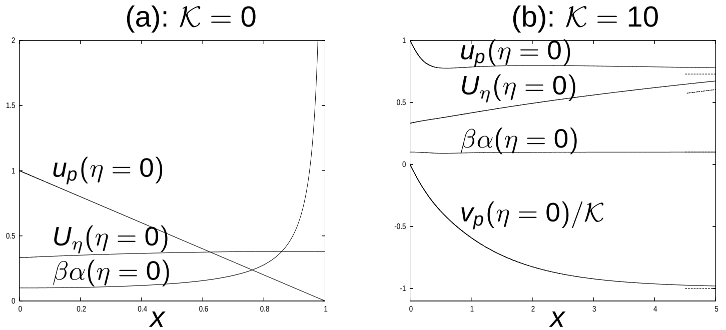
Edge results



Development of the edge quantities (a) u_{pe} and (b) α_e; solid $m = .50$; dashed $m = .211$; dotted $m = .10$, all with $\beta/\gamma = 1$; $\mathcal{K}$ not relevant here.

Can show $\alpha_e \sim \frac{1}{x - x_0}$ if $m > m_{crit}$ - violates $\alpha << 1$

BOUNDARY-LAYER RESULTS



Development of wall values with x for $m = 0$, $\beta/\gamma = 1$. In case (b), leading-order asymptotic forms for $x \to \infty$ are shown for $x > 4.5$;

Singularities inside the boundary layer

Taking $\mathcal{K} = 0$ (for simplicity - particle flow can be solved explicitly on $y = 0$):

$$\alpha_w = \frac{\alpha_0}{U_{pw}} = \frac{\alpha_0}{1 - \frac{\beta x}{\gamma}}, \quad U_{pw} > 0.$$

Therefore volume fraction is singular part way along wall, and so model breaks down (Wang & Glass, 1988 continued their computation through the singularity).
Can be a ‘race’ between inner and outer singularities.

$\mathcal{K} < 0$

If gravitational forces act away from the wall, close to wall characteristics directed outwards; local analysis (as $x \to 0$) reveals a discontinuity in α along $y = y_{crit} = -\mathcal{K} \cos\theta / U_{p0}$ where $u_p = U_{p0} + \ldots$:
for $y < y_{crit}$, $\alpha = 0$ (particle free), for $y > y_{crit}$, $\alpha = \alpha_0$

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- Computations and analyses inform each other
- Contaminants can have a profound effect!