Hyperbolize it.

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Sushi is Diffusion

A harmony of rice and topping:
rice crumbles and dissolves
with topping in your mouth.
- Tasty diffusion

http://www.hanayuubou.com/

Years of training required to make such good sushi.
Know-how is in experts' hands.

The same for diffusion schemes,
but the development is still in early stage.

Lack of Guiding Principle

Guiding Principles

Upwind for advection - hyperbolic

A variety of schemes generated:

Flux-Vector/Difference Splitting
Multidimensional upwind - RD, FS schemes
Riemann Solvers, CUSP, AUFS, AUSM, LDFSS,
HLL, Steger-Warming, SUPG, etc.

Isotropic for diffusion - parabolic ?

Not that successful, especially for unstructured and high-order.
What can be a useful guiding principle for diffusion?

Behold, we already have it.

Algorithm Research Continues

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{B}\mathbf{U}_{xx} + \mathbf{C}\mathbf{U}_{xxx} + \cdots + \mathbf{S}$$

	Hyperbolic	Parabolic	Dispersion	Source
Algorithm	Well developed	No so well	Not so well	Tricky

Really? It is already well developed for all terms if we ...

Hyperbolize Them

First-Order Hyperbolic System Method

JCP2007, 2010, 2012, AIAA2009, 2010, 2011, 2013, CF2011

$$\mathbf{U}_t + \mathbf{A}\mathbf{U}_x = \mathbf{B}\mathbf{U}_{xx} + \mathbf{C}\mathbf{U}_{xxx} + \cdots + \mathbf{S}$$

$$\Downarrow$$

$$\tilde{\mathbf{W}}_t + \tilde{\mathbf{A}}\tilde{\mathbf{W}}_x = 0$$

Dramatic simplification/improvements to numerical methods

Methods for hyperbolic system applicable to all terms

Turn Every Food into a Burger!

Simple, Efficient, Accurate.

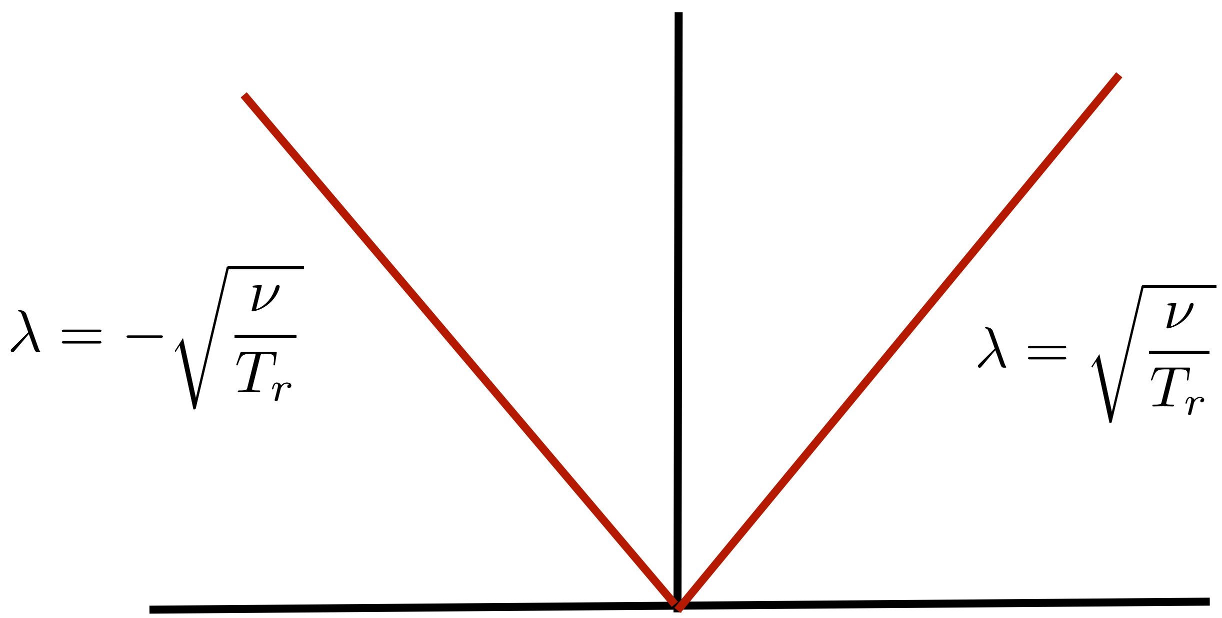
Sushi Burger!

It looks eccentric, but the taste is the same, or even better!

Hyperbolic Diffusion System

$$\begin{aligned} u_t &= \nu\, p_x \\ p_t &= (u_x - p)/T_r \end{aligned}$$

This is hyperbolic, describing a symmetric wave:



Equivalent to diffusion eq. in the steady state for any Tr.

$\longrightarrow$ *Tr is a free parameter.*

Hyperbolic Diffusion Scheme

Upwind scheme for diffusion: $T_r = O(1)$ JCP2007, 2010

$$\frac{d\mathbf{U}_j}{dt} = -\frac{1}{h}\left[\mathbf{F}_{j+1/2} - \mathbf{F}_{j-1/2}\right] + \mathbf{Q}_j$$

$$\mathbf{F}_{j+1/2} = \underbrace{\frac{1}{2}\left[\mathbf{F}_R + \mathbf{F}_L\right]}_{\textbf{Consistent}} - \underbrace{\frac{1}{2}|\mathbf{A}|\left(\mathbf{U}_R - \mathbf{U}_L\right)}_{\textbf{Dissipation}}$$

Rapid steady convergence with O(h) time step, not O(h^2).

Damped scheme for diffusion: $T_r = O(h^2)$ JCP2007

Strong damping with O(h^2) time step.

Same order of accuracy for solution and gradient.

Traditional Diffusion Scheme

Scalar diffusion scheme can be derived from hyperbolic scheme.

Traditional Diffusion Scheme:

Simply ignore the second equation in hyperbolic scheme, and reconstruct the gradients.

$$\frac{du_j}{dt} = -\frac{1}{h}\left[f_{j+1/2} - f_{j-1/2}\right]$$

$$f_{j+1/2} = \underbrace{\frac{1}{2}\left[\nu(u_x)_j + \nu(u_x)_{j+1}\right]}_{\textbf{Consistent}} + \underbrace{\frac{\nu\alpha}{2h}(u_R - u_L)}_{\textbf{Damping (from dissipation)}}$$

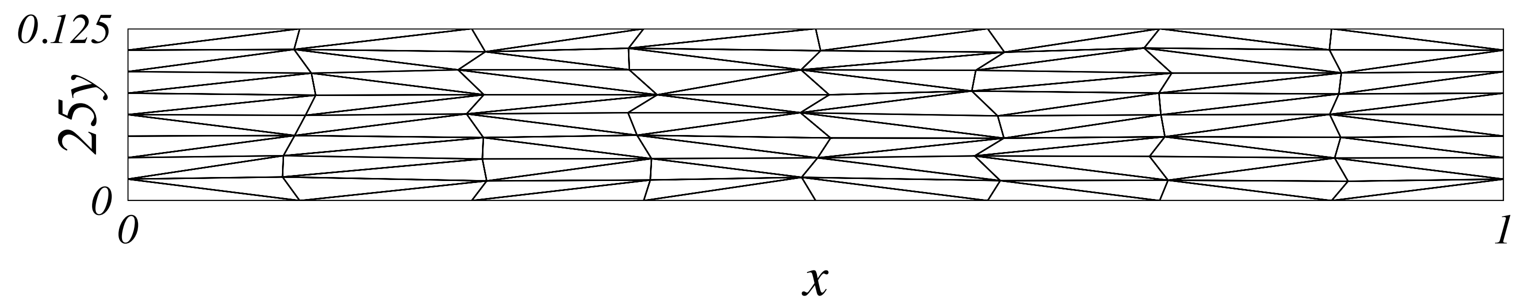
High-frequency damping term is introduced automatically.
It is essential for accuracy and robustness.

See AIAA2010, CF2011

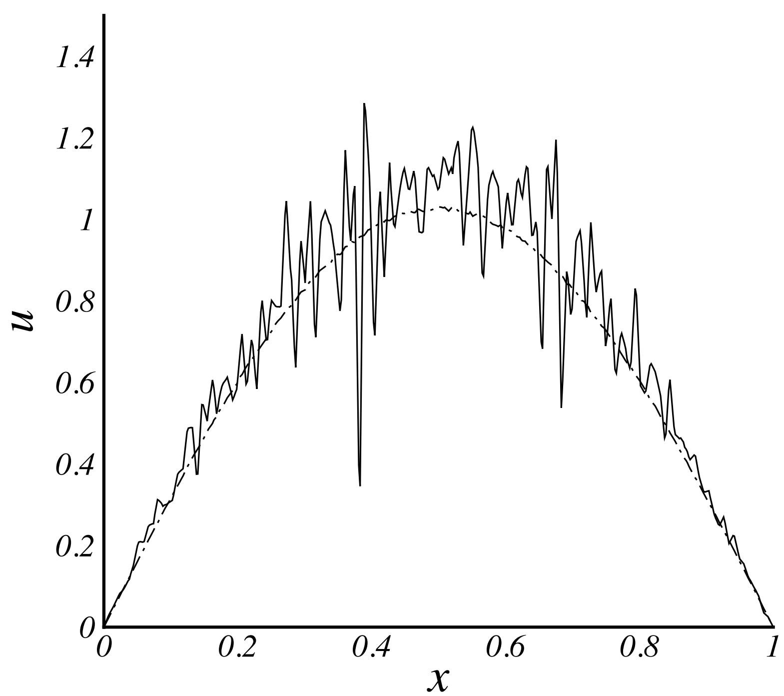
For every advection scheme,
there is a corresponding diffusion scheme.

Damping is Essential

Highly-skewed unstructured grid (unsteady diffusion problem)

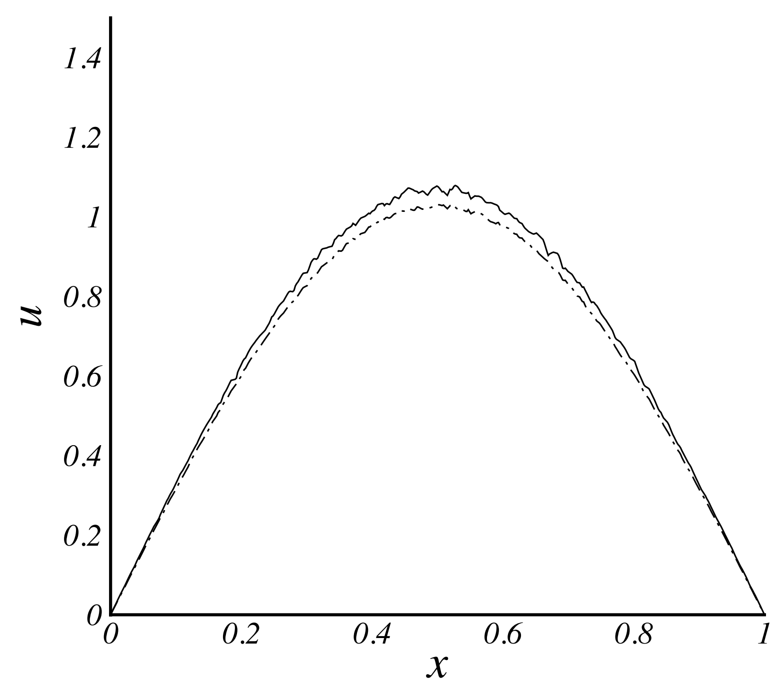


Results for cell-centered finite-volume method

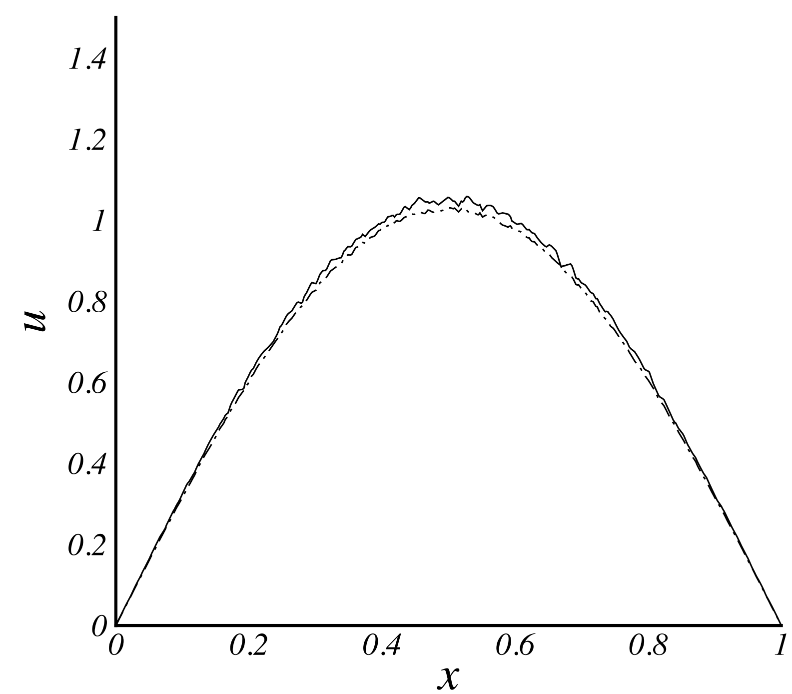


Avg-LSQ Edge-normal

$\alpha \approx 0$



$\alpha = 1$

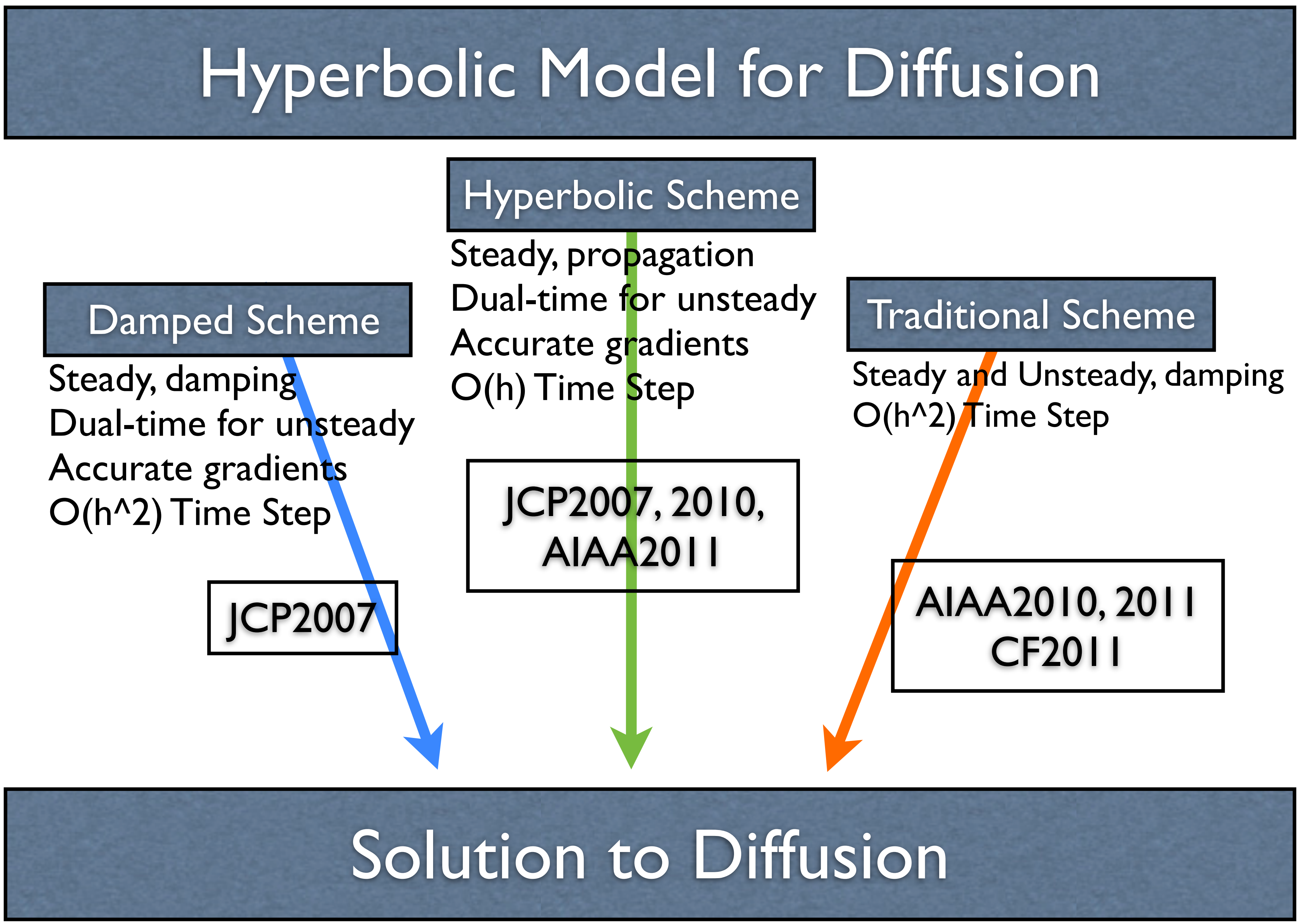


$\alpha = 8/3$

Damping term is critical for unstructured computations

See AIAA2010, CF2011, for many examples.

Three Paths to Take

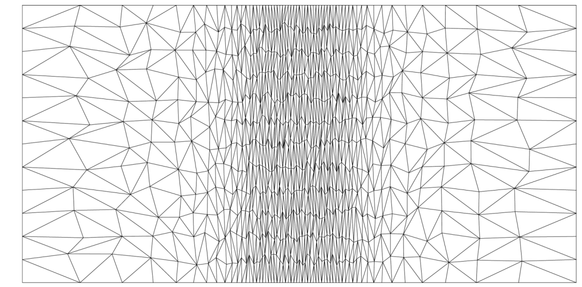


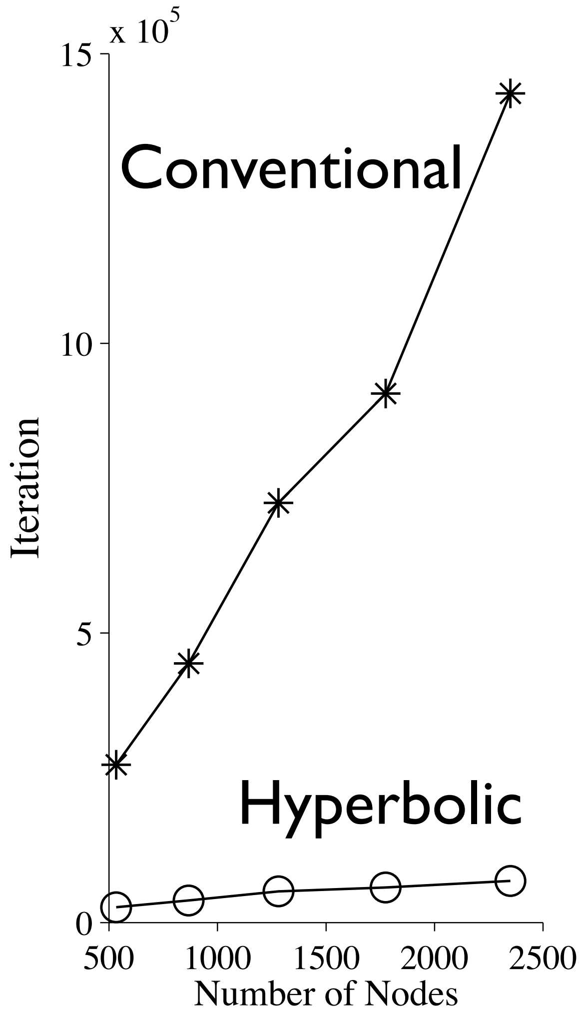
It all starts from the discretization of the hyperbolic model.

Navier-Stokes Results

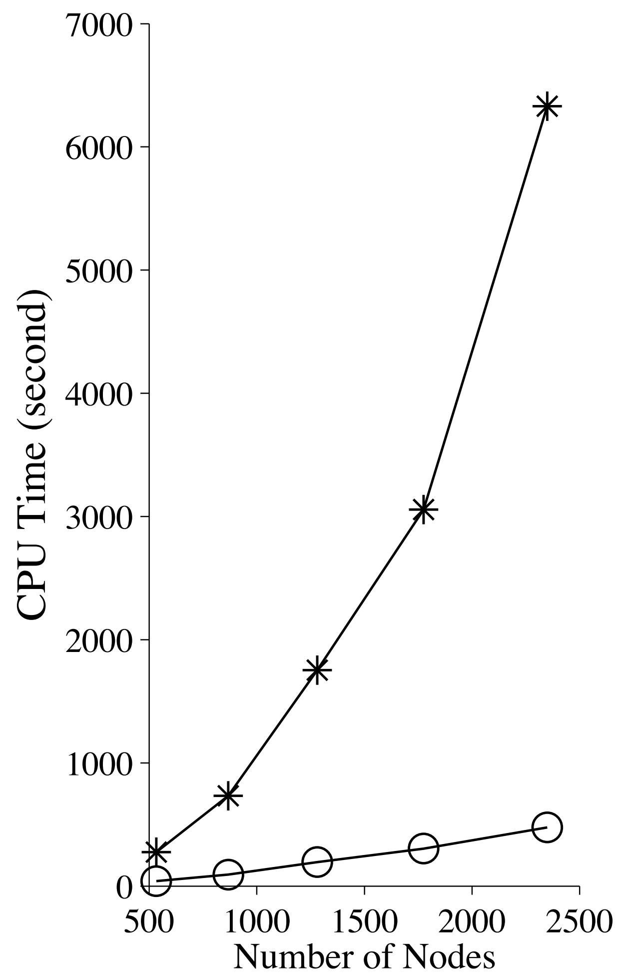
2nd-order finite-volume schemes

Viscous Shock-Structure Problem

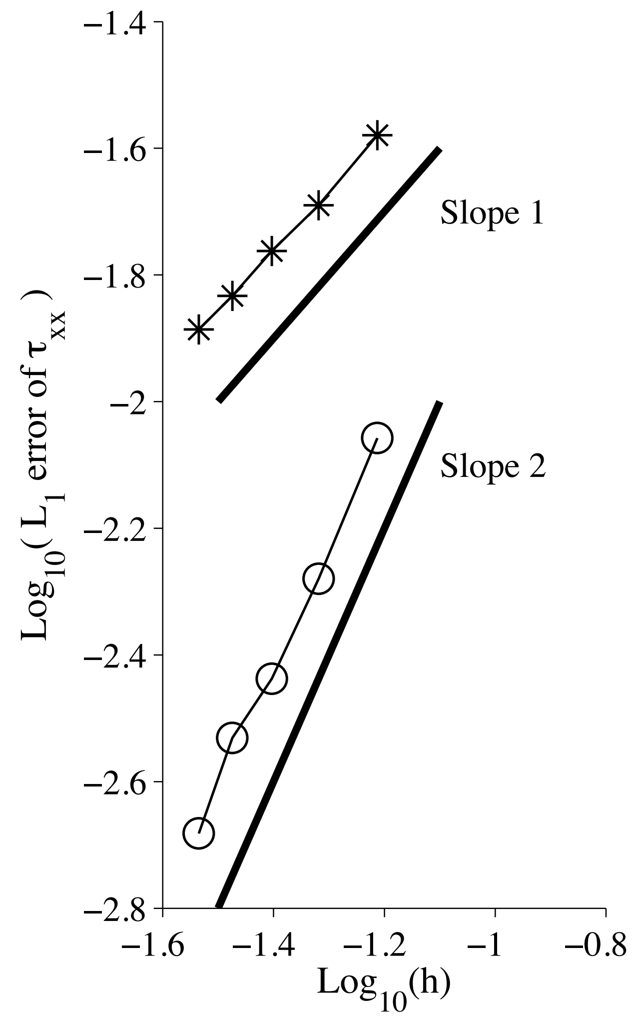




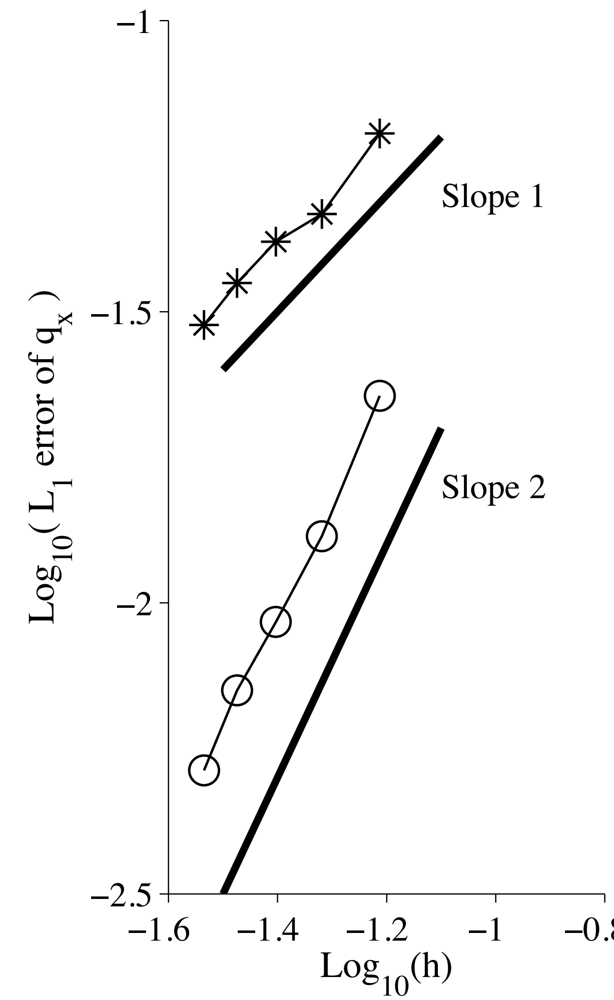
Iteration



CPU Time



Viscous Stress



Heat Flux

The idea extended to nonlinear system.

Hyperbolize to the Future

Construct a hyperbolic system and discretize it:

$$\tilde{\mathbf{W}}_t + \tilde{\mathbf{A}}\tilde{\mathbf{W}}_x = 0$$

Hitherto unexpected advantages being discovered:

Higher order accuracy for viscous/heat fluxes
Orders of magnitude faster viscous computation by O(h) time step
Compact stencil for high-order derivatives
High-order advection schemes for diffusion
Boundary conditions made simple (all Dirichlet; local characteristics)
A greater variety of viscous discretizations
Damping term incorporated automatically into viscous schemes

Large eccentricity leads to a hyperbolic trajectory, which enables us to escape towards the future.